OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2612**

Mechanics 6

Friday **24 JUNE 2005** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8$ m s^{-2} unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

1 *Option 1: Rotation of a Rigid Body*

A snooker ball is a uniform sphere of mass m and radius a. It rests on a horizontal table. The coefficient of friction between the ball and the table is μ. The ball is struck with a cue which exerts a horizontal impulse of magnitude J at a point $\frac{1}{2}a$ above the table. The line of action of the impulse is in a vertical plane through the centre of the ball. After the ball is struck, it has initial speed u.

(i) Show that the initial angular speed is $\dfrac{5u}{4a}$. Deduce that slipping occurs. [6]

(ii) Find expressions for both the velocity and the angular velocity of the sphere at time t after the ball is struck. [7]

(iii) Find the time at which the ball stops slipping. [3]

(iv) The sphere then rolls without slipping. Show that it moves with constant velocity and calculate this velocity. Find also the magnitude of the frictional force, justifying your answer. [4]

2 *Option 2: Vectors*

(a) Forces $\begin{pmatrix}-1\\2\\-1\end{pmatrix}$, $\begin{pmatrix}2\\-1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ act at points with position vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\2\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\\lambda\end{pmatrix}$ respectively.

Show that the forces reduce to a couple. Find the value of λ for which the magnitude of the couple is minimised. What is this minimum value? [9]

(b) The position vector of a particle P of mass m at time t is

$$\mathbf{r} = \begin{pmatrix}2\cos\omega t\\-\sin\omega t\\2t\end{pmatrix},$$

where ω is a positive constant.

(i) Calculate the angular momentum of P about the origin. [6]

(ii) Hence find the torque about O acting on the particle. Show that for $t > 0$ the torque is never zero. [5]

3 *Option 3: Stability and Oscillations*

A rigid circular hoop of radius a is fixed in a vertical plane. A light elastic string of natural length a and modulus λ is attached to the highest point of the hoop. The other end of the string is attached to a small, smooth ring of mass m which is threaded onto the hoop. When the system is in equilibrium, the angle the string subtends at the centre of the circle is θ, as shown in Fig. 3.

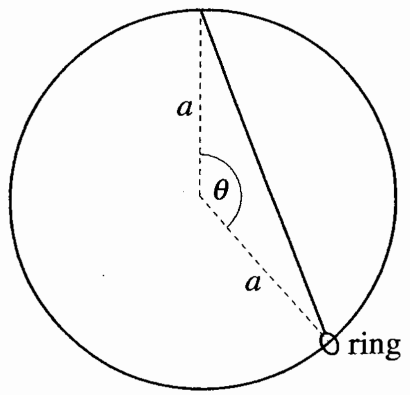


Fig. 3

(i) Find the potential energy, V, of the system in terms of θ, and show that

$$\frac{dV}{d\theta} = \lambda a \cos\tfrac{1}{2}\theta\left(-1+2\left(1-\frac{mg}{\lambda}\right)\sin\tfrac{1}{2}\theta\right).$$ [6]

(ii) Show that there is a position of equilibrium at $\theta = \pi$. Show further that it is stable if $\lambda < 2mg$. Investigate the stability when $\lambda > 2mg$ and when $\lambda = 2mg$. [8]

(iii) Show that if $\lambda > 2mg$ there is also a position of equilibrium for $\theta < \pi$ and that if $\lambda \leqslant 2mg$ the only equilibrium position is at $\theta = \pi$. [6]

4 *Option 4: Variable mass*

A rocket in deep space is heading for a space station. Gravitational forces may be neglected. The initial mass of the rocket is m_0 and the propulsion system ejects matter at a mass rate k with speed u relative to the rocket. The rocket starts from rest at a distance a from the space station and travels in a straight line towards it. At time t the velocity of the rocket is v. The rocket ejects a total mass of $\frac{1}{2}m_0$.

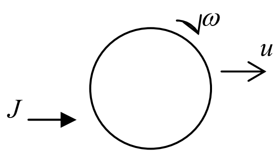
(i) Show that $(m_0 - kt)\dfrac{dv}{dt} = uk$ and hence find an expression for v at time t. [8]

(ii) Show that the distance travelled while matter is being ejected is $\dfrac{um_0}{2k}(1 - \ln 2)$. [8]

[You may use the result $\int \ln x \, dx = x\ln x - x$ (+ constant) without proof.]

(iii) If $a = \dfrac{um_0}{2k}$ show that the total time taken to reach the space station is $\dfrac{m_0}{k}$. [4]

Mark Scheme 2612
June 2005

1(i)		$J = mu$	B1	
		$-J \cdot \frac{1}{2}a = \frac{2}{5}ma^2\omega$	M1	
			A1	
		$-\frac{1}{2}mau = \frac{2}{5}ma^2\omega$	M1	eliminate J, m
		$\omega = -\dfrac{5u}{4a}$ so ang.speed $= \dfrac{5u}{4a}$	E1	
	$u \neq a\omega \Rightarrow$ slipping occurs		E1	
				6
(ii)	$m\ddot{x} = -F$		M1	N2L
	$= -\mu mg$		B1	$F = \mu mg$ stated or used
	$\dot{x} = u - \mu gt$		A1	
	$\frac{2}{5}ma^2\ddot{\theta} = aF = \mu mga$		M1	
			A1	equation of rotation (ignore sign)
	$\Rightarrow \ddot{\theta} = \dfrac{5\mu g}{2a}$		M1	
	$\dot{\theta} = -\dfrac{5u}{4a} + \dfrac{5\mu g}{2a}t$		A1	cao
				7
(iii)	stops slipping when $\dot{x} = a\dot{\theta}$		B1	
	$u - \mu gt = -\dfrac{5u}{4} + \dfrac{5\mu g}{2}t$		M1	use condition
	$t = \dfrac{9u}{14\mu g}$		A1	cao
				3
(iv)	$\dot{x} = a\dot{\theta} \Rightarrow \ddot{x} = a\ddot{\theta}$		M1	
	$F \neq 0 \Rightarrow \ddot{x}$ and $\ddot{\theta}$ have opposite signs so $\ddot{x} = \ddot{\theta} = 0$		E1	
	therefore constant velocity of $u - \mu g\left(\frac{9u}{14\mu g}\right) = \frac{5}{14}u$		F1	follow their t in either $\dot{x}$ or $a\dot{\theta}$
	$F = -m\ddot{x} = 0$		B1	must be justified
				4

2(a)	$\Sigma\mathbf{F} = \begin{pmatrix}-1\\2\\-1\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix} + \begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$	M1 A1	evidence of working needed	
	$\mathbf{C} = \Sigma\mathbf{r}\times\mathbf{F}$	M1		
	$= \begin{vmatrix}\mathbf{i} & 1 & -1\\ \mathbf{j} & 1 & 2\\ \mathbf{k} & 0 & -1\end{vmatrix} + \begin{vmatrix}\mathbf{i} & 0 & 2\\ \mathbf{j} & 2 & -1\\ \mathbf{k} & 0 & 0\end{vmatrix} + \begin{vmatrix}\mathbf{i} & 0 & -1\\ \mathbf{j} & 1 & -1\\ \mathbf{k} & \lambda & 1\end{vmatrix}$	M1	attempt vector product	
	$= \begin{pmatrix}-1\\1\\3\end{pmatrix} + \begin{pmatrix}0\\0\\-4\end{pmatrix} + \begin{pmatrix}\lambda+1\\-\lambda\\1\end{pmatrix} = \begin{pmatrix}\lambda\\1-\lambda\\0\end{pmatrix}$	A1		
	which cannot be zero hence a couple	E1	must observe that **C** is non-zero	
	$\lvert\mathbf{C}\rvert^2 = \lambda^2 + (1-\lambda)^2 = 2\left(\lambda - \frac{1}{2}\right)^2 + \frac{1}{2}$	M1	or alternative method to find minimum	
	hence minimum magnitude of couple = $\frac{1}{\sqrt{2}}$	A1	must be magnitude	
	when $\lambda = \frac{1}{2}$	A1		9
(b)(i)	$\mathbf{v} = \begin{pmatrix}-2\omega\sin\omega t\\ -\omega\cos\omega t\\ 2\end{pmatrix}$	M1 A1	differentiate **r**	
	$\mathbf{L} = \mathbf{r}\times m\mathbf{v}$	M1		
	$= m\begin{pmatrix}-2\sin\omega t + 2\omega t\cos\omega t\\ -4\omega t\sin\omega t - 4\cos\omega t\\ -2\omega\end{pmatrix}$	M1 A1 A1	attempt vector product one correct component all correct (aef)	6
(ii)	torque $= \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$	M1		
	$= m\begin{pmatrix}-2\omega\cos\omega t + 2\omega\cos\omega t - 2\omega^2 t\sin\omega t\\ -4\omega^2 t\cos\omega t - 4\omega\sin\omega t + 4\omega\sin\omega t\\ 0\end{pmatrix}$	M1	differentiate	
	$= -2m\omega^2 t\begin{pmatrix}\sin\omega t\\ 2\cos\omega t\\ 0\end{pmatrix}$	A1	aef	
	if $t > 0$, torque = 0 $\Rightarrow \sin\omega t = \cos\omega t = 0$ but $\sin\omega t = 0 \Rightarrow \cos\omega t = \pm 1$ so torque not zero	M1 E1	attempt to show vector non-zero convincing argument	5

3(i)	$V = mga\cos\theta + \frac{\lambda}{2a}\left(2a\sin\frac{1}{2}\theta - a\right)^2$	M1 A1 A1	attempt V in terms of θ GPE ($\pm$ constant) EPE
	$V'(\theta) = -mga\sin\theta + \frac{\lambda}{2a}\cdot 2\left(2a\sin\frac{1}{2}\theta - a\right)\left(a\cos\frac{1}{2}\theta\right)$	M1 M1	differentiate good attempt at both terms
	$= -2mga\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta + \lambda a\cos\frac{1}{2}\theta\left(2\sin\frac{1}{2}\theta - 1\right)$		
	$= \lambda a\cos\frac{1}{2}\theta\left(-1 + 2\left(1 - \frac{mg}{\lambda}\right)\sin\frac{1}{2}\theta\right)$	E1	
			6
(ii)	$\theta = \pi \Rightarrow \cos\frac{1}{2}\theta = 0 \Rightarrow V'(\theta) = 0$	M1	
	$\Rightarrow$ equilibrium	E1	
	$V''(\theta) = -\frac{1}{2}\lambda a\sin\frac{1}{2}\theta\left(-1 + 2\left(1 - \frac{mg}{\lambda}\right)\sin\frac{1}{2}\theta\right)$ $+\lambda a\cos\frac{1}{2}\theta\cdot\left(1 - \frac{mg}{\lambda}\right)\cos\frac{1}{2}\theta$	M1	differentiate again
	$V''(\pi) = -\frac{1}{2}\lambda a\left(1 - \frac{2mg}{\lambda}\right)$		
	$\lambda < 2mg \Rightarrow V''(\pi) > 0 \Rightarrow$ stable	M1 E1	consider sign of $V''(\pi)$
	$\lambda > 2mg \Rightarrow V''(\pi) < 0 \Rightarrow$ unstable	B1	
	$\lambda = 2mg \Rightarrow V'(\theta) = 2mga\cos\frac{1}{2}\theta(-1 + \sin\frac{1}{2}\theta)$		
	$V'(\pi - \varepsilon) = (+)(-) < 0, \quad V'(\pi + \varepsilon) = (-)(-) > 0$	M1	any valid method
	$\Rightarrow$ minimum, hence stable	A1	
			8
(iii)	$\cos\frac{1}{2}\theta \neq 0, V'(\theta) = 0 \Rightarrow -1 + 2\left(1 - \frac{mg}{\lambda}\right)\sin\frac{1}{2}\theta = 0$	M1	
	$\Rightarrow \sin\frac{1}{2}\theta = \frac{1}{2\left(1 - \frac{mg}{\lambda}\right)}$	A1	or equivalent
	$\lambda > 2mg \Rightarrow 0 < \frac{mg}{\lambda} < \frac{1}{2} \Rightarrow \left(\frac{1}{2} <\right)\sin\frac{1}{2}\theta < 1$	M1	only required to establish $\sin\frac{1}{2}\theta < 1$
	$\Rightarrow \theta < \pi$	E1	
	$\lambda = 2mg \Rightarrow \sin\frac{1}{2}\theta = 1 \Rightarrow \theta = \pi$ as before	E1	
	$\lambda < 2mg \Rightarrow \sin\frac{1}{2}\theta > 1 \Rightarrow$ no solutions, so only $\theta = \pi$ as before	E1	
			6

4(i)	If δm is mass lost in time δt PCLM $mv = (m-\delta m)(v+\delta v) - \delta m(u-v)$	M1	change in momentum over time δt	
	$m\frac{\delta v}{\delta t} = u\frac{\delta m}{\delta t} + \delta v\frac{\delta m}{\delta t} \Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t} = -u\frac{\mathrm{d}m}{\mathrm{d}t}$ $\left(\text{NB using } \delta m > 0 \text{ but } \frac{\mathrm{d}m}{\mathrm{d}t} < 0\right)$	A1	accept sign error	
	$\frac{\mathrm{d}m}{\mathrm{d}t} = -k \Rightarrow m = m_0 - kt$	M1	get m in terms of t	
	$\Rightarrow (m_0 - kt)\frac{\mathrm{d}v}{\mathrm{d}t} = uk$	E1		
	$v = \int \frac{uk}{m_0 - kt}\mathrm{d}t$	M1	separate and integrate	
	$= -u\ln(m_0 - kt) + c$	A1	multiple of $u\ln(m_0 - kt)$	
	$t = 0, v = 0 \Rightarrow c = u\ln m_0$	M1	use initial condition	
	$v = u\ln\left(\frac{m_0}{m_0 - kt}\right)$	A1	aef	
				8
(ii)	matter all ejected when $kt = \frac{1}{2}m_0$	M1		
	$t = \frac{m_0}{2k}$	A1	cao	
	distance = $\int_0^{\frac{m_0}{2k}} u\ln\left(\frac{m_0}{m_0 - kt}\right)\mathrm{d}t$	M1 M1	integral limits (0 to their t)	
	$= \int_0^{\frac{m_0}{2k}} -u\ln\left(1 - \frac{k}{m_0}t\right)\mathrm{d}t$	M1	rearrange into any suitable form for integrating	
	$= \frac{um_0}{k}\left[\left(1-\frac{k}{m_0}t\right)\ln\left(1-\frac{k}{m_0}t\right) - \left(1-\frac{k}{m_0}t\right)\right]_0^{\frac{m_0}{2k}}$	M1 A1	reasonable attempt at integral	
	$= \frac{um_0}{k}\left(\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2} - -1\right) = \frac{um_0}{2k}(1 - \ln 2)$	E1		
				8
(iii)	speed when fuel runs out $= u\ln\left(\frac{m_0}{\frac{1}{2}m_0}\right) = u\ln 2$	M1	use $kt = \frac{1}{2}m_0$ or their t from (ii) in their v	
	distance remaining $= a - \frac{um_0}{2k}(1 - \ln 2) = \frac{um_0}{2k}\ln 2$	B1		
	time after fuel runs out $\frac{\frac{um_0}{2k}\ln 2}{u\ln 2} = \frac{m_0}{2k}$	M1	their distance/their speed	
	total time $= \frac{m_0}{2k} + \frac{m_0}{2k} = \frac{m_0}{k}$	E1	must be all correct	
				4

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General Comments

Questions one was not a popular choice, with most candidates attempting the other three questions. The standard of work varied widely, but most candidates were able to show some competence at three questions.

Comments on Individual Questions

1) (i) Most candidates were able to find the angular speed and deduce that slipping occurred.

(ii) The velocity and angular velocity expressions were often well done, although some confusion of signs occurred with the angular velocity.

(iii) Most candidates knew the condition for slipping to stop.

(iv) Most candidates assumed that the frictional force was zero without any explanation, and then used this to show that the velocity was constant.

2) (a) This part was generally done well, except few candidates pointed out that the total moment was not zero, and hence this was a couple (rather than equilibrium).

(b)(i) There were many good solutions, but some tried to work with scalars rather than vectors.

(ii) The word ‘hence’ indicated that candidates should use the angular momentum, which some did not. However, there were many good solutions to this part.

3) (i) Most candidates knew what to do here, but algebraic slips were common. Some candidates made very heavy weather of finding the gravitational potential energy, using very complicated geometry.

(ii) This was often well done except for the $\lambda = 2mg$ case. In this case it was very surprising how many candidates wrongly thought that a zero second derivative guaranteed a point of inflection. Also surprising was that they often then deduced that the equilibrium was stable on one side and unstable on the other!

(iii) Again, most candidates knew what to do, but algebraic errors hindered some.

4) (i) Deriving the relevant differential equation was often done well, but some candidates confused the signs. Most candidates found v correctly.

(ii) Most candidates knew how to find the distance, but many struggled with the integral, even with the result that was given in the question.

(iii) Some candidates produced excellent concise solutions to this part, but some thought that integration was required.